State-dependent fire models and related renewal processes

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We introduce a general class of stochastic processes forced by instantaneous random fires (i.e., jumps) that reset the state variable x to a given value. Since in many physical systems the fire activity is often dependent on the actual value of the state variable, as in the case of natural fires in ecosystems and firing dynamics in neuronal activity, the frequency of fire occurrence is assumed to be state dependent. Such dynamics leads to independent interfire statistics—i.e., to renewal point processes. Various functions relating the frequency of fire occurrence to $x(t)$ are analyzed and compared. The relation between the probabilistic dynamics of $x(t)$ and the interfire statistics is derived and some exact probability distribution of both $x(t)$ and the interfire times are obtained for systems with different degrees of complexity. After studying processes in which the fire activity is coupled only to a deterministic drift, we also analyze processes forced by either additive or multiplicative Gaussian white noise.

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I. INTRODUCTION

Models with state-dependent noise have been widely studied in systems driven by Gaussian noise—that is, in the case of multiplicative noise [1–7]. On the other hand, the role of the state dependence for other types of noise is less clear and only a few studies have been carried out [8,9].

In the present paper we deal with state-dependent fire models—i.e., models describing processes with instantaneous jumps that reinitialize the state variable to a given value. These models find multiple applications in different disciplines that often deal with problems related to renewal theory [10]—that is, processes in which the interfire or renewal times are independent and identically distributed random variables. The applications range from analysis of failure and replacement of components to queuing processes, such as traffic flow and electronic counter [10,11], to storage problems [12] and population dynamics [13].

Fire models are also largely adopted to describe neural signal transmission [14–18], such as the leaky integrate-and-fire model [18,19], which assumes the voltage across a nerve membrane between successive neuron excitations to behave according to an Ornstein-Uhlenbeck process. When the voltage reaches a certain threshold, an excitation occurs, which resets the voltage to a lower value. The interspike statistics depends on the dynamics of both the voltage and threshold, which might not be constant [17] and depend on the actual value of the voltage. Hence, such dynamics can generate times of neuron excitations that are points of a state-dependent renewal process.

Another example of application of state-dependent fire models leading to a renewal process can be found in the modeling of disturbances and catastrophic events in ecology [13], with applications to fire dynamics in ecosystems [9,20,21]. Natural fires represent an important determinant of above-ground biomass, and in some ecosystems they are considered a necessary component for their preservation [22]. The occurrence of fires, although random, is at the same time strongly dependent on the dynamics of the above-ground biomass, which represents the fuel. Thus, the probability of fire occurrence typically increases as the above-ground biomass production grows, while, after a fire, the biomass is reduced and the fire probability is reduced until biomass grows again.

One of the main properties characterizing fire processes is the interarrival time of fires. We introduce an approach to find the distribution of the time between successive fires that differs and is mathematically more advantageous than the more common approach based on finding the first-passage rates through a threshold [14–16].

After presenting the general mathematical framework used to describe the state-dependent fire processes and to find the interfire statistics (Sec. II), Secs. III and IV show applications to not-diffusive and diffusive systems, respectively.

II. MATHEMATICAL MODEL

A simplified way to model processes such as those discussed before is to describe the system in terms of a single representative stochastic variable $x(t)$, the trajectory of which is perturbed by fires that reset the state of the system to a given value, hereinafter assumed to be zero without loss of generality. The dynamics of $x(t)$ between two fires is assumed to be driven by a deterministic component plus a random forcing: namely, a multiplicative white Gaussian noise either additive or multiplicative. The fires or renewal events can be modeled as a Poisson process $F(x,t)$, with frequency λ, which in general may be time and state dependent—i.e., $\lambda=\lambda(x(t),t)$. Using the terminology of renewal theory, the function $\lambda(x(t),t)$ represents the *age-specific failure rate* [10]—i.e., the probability that, being at a time (age) t, a fire is released in the infinitesimal time

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interval $t+dt$. Accordingly, the equation for $x(t)$ can be written as

$$\frac{dx}{dt}=f(x)+g(x)\xi(t)-F(x,t), \tag{1}$$

where $f(x)$ and $g(x)$ are deterministic functions and $\xi(t)$ is white Gaussian noise, with $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(s)\rangle=2\sigma^2\delta(t-s)$, σ^2 being the noise strength. The system therefore starts from a given initial value $x(0)=x_0$, evolves along a trajectory up to a certain time at which a fire is released, and then restarts from zero. After the first reinitialization, the fires generate a renewal process, since interfire times are independent and identically distributed random variables [10]. Because of the state dependence of fire occurrence, the distribution of the first-fire times depends on the initial condition x_0 and, unless $x_0=0$, it is different from that of the times between subsequent fires—that is, when the system restarts from $x(t)=0$. This means that the processes generated by state-dependent fire models are modified renewal processes when $x_0\neq 0$ and become ordinary renewal processes only when $x_0=0$ [23].

When Eq. (1) is interpreted in the Itô sense, the transition probability distribution function of $x(t)$, $p(x,t)$, satisfies the forward master equation [23–25]

$$\frac{\partial}{\partial t}p(x,t)=-\frac{\partial}{\partial x}[f(x)p(x,t)]+\sigma^2\frac{\partial^2}{\partial x^2}[g^2(x)p(x,t)] - \lambda(x,t)p(x,t)+\delta(x)\int_\Omega \lambda(z,t)p(z,t)dz, \tag{2}$$

where Ω is the domain of x. Equation (2) states that the time variation of $p(x,t)$ is given by a contribution related to the drift $f(x)$, a diffusive term generated by the Gaussian noise, a term that defines the loss of probability due to the jumps, and an integral term that represents an injection of mass in $x=0$ due to the renewal events. This last contribution, which is simply $\delta(x)\langle\lambda(t)\rangle$, where $\langle\lambda(t)\rangle$ is the mean frequency of the fire occurrences, maintains the area of the probability distribution function (PDF) constant, reintroducing in $x=0$ the mass lost through the jumps [i.e., the third term in the right-hand side (RHS) of Eq. (2)]. Equation (2) is solved assuming the initial condition $p(x,0)=\delta(x-x_0)$ and the integral $\int_\Omega p(x,t)dx$ is constant in time and equal to 1. In those cases in which the transient solution of Eq. (2) cannot be found analytically, we will study, when possible, the stationary solution $p(x)$, which is the PDF reached by the system in the long time, when $\partial_t p(x,t)=0$.

In order to determine analytically the distribution of interfire times, it is useful to introduce a slightly modified version of Eq. (2), in which the integral term is eliminated. The corresponding process is identical to the previous one until the occurrence of the first fire, after which the trajectory is not renewed and the process stops. The related PDF, indicated as $p_\lambda(x,t)$, satisfies the equation

$$\frac{\partial}{\partial t}p_\lambda(x,t)=-\frac{\partial}{\partial x}[f(x)p_\lambda(x,t)]+\sigma^2\frac{\partial^2}{\partial x^2}[g^2(x)p_\lambda(x,t)] - \lambda(x,t)p_\lambda(x,t), \tag{3}$$

with initial condition $p_\lambda(x,0)=\delta(x-x_0)$. The solution $p_\lambda(x,t)$ loses mass in time at the rate $\int_\Omega\lambda(x,t)p_\lambda(x,t)$, so that its area tends to zero as t goes to $+\infty$, providing the fraction of trajectories that survive without a fire (i.e., renewal event) up to time t. Thus, interestingly, the PDF $p_\lambda(x,\tau)$ is related to the probability that the process $x(t)$ has not experienced a jump up to time $t=\tau$, $\mathcal{F}(\tau)$, by

$$\mathcal{F}(\tau)=\int_\Omega p_\lambda(x,\tau)dx, \tag{4}$$

which is commonly referred to as *survivor function* [10,21]. From this, it follows that the PDF of the time of first-fire occurrence, $p_\tau(\tau)$, is

$$p_\tau(\tau)=-\frac{d}{d\tau}\mathcal{F}(\tau), \tag{5}$$

which depends on the initial condition x_0. When $x_0=0$, $p_\tau(\tau)$ also represents the PDF of the interfire times, since after the first fire the system always restarts from $x(t)=0$.

The PDF $p_\tau(\tau)$ can be evaluated directly from Eq. (3), provided that both $f(x)p_\lambda(x,\tau)$ and $\partial_x[g(x)^2p_\lambda(x,\tau)]$ go to zero at the boundaries of Ω. In fact, integrating both sides with respect to x over the entire domain yields

$$p_\tau(\tau)=-\frac{d}{d\tau}\int_\Omega p_\lambda(x,\tau)dx=\int_\Omega \lambda(x)p_\lambda(x,\tau)dx. \tag{6}$$

The adoption of Eq. (6) is typically not very useful, since Eq. (3) is not solvable in the majority of the cases. However, since the Laplace transform of $p_\lambda(x,\tau)$, $p_\lambda^*(u,\tau)$, is usually more easily obtained, the survivor function can be also evaluated as

$$\mathcal{F}(\tau)=\left.p_\lambda^*(u,\tau)\right|_{u=0}=\left.\int_{-\infty}^{+\infty}e^{-ux}p_\lambda(x,\tau)dx\right|_{u=0}. \tag{7}$$

A first simple example of the use of Eq. (6) can be obtained when λ does not depend on $x(t)$—i.e., $\lambda=\lambda(t)$. In this case, with the substitution $p_\lambda=\tilde{p}_\lambda\exp[-\int_0^\tau\lambda(u)du]$, Eq. (3) becomes

$$\frac{\partial}{\partial \tau}\tilde{p}_\lambda(x,\tau)=-\frac{\partial}{\partial x}[f(x)\tilde{p}_\lambda(x,\tau)]+\sigma^2\frac{\partial^2}{\partial x^2}[g^2(x)\tilde{p}_\lambda(x,\tau)], \tag{8}$$

the solution of which satisfies the condition $\int_\Omega\tilde{p}_\lambda dx=1$. Thus Eq. (6) gives

$$p_\tau(\tau)=\lambda(\tau)\int_\Omega\tilde{p}_\lambda\exp\left[-\int_0^\tau\lambda(u)du\right]dx =\lambda(\tau)\exp\left[-\int_0^\tau\lambda(u)du\right], \tag{9}$$

which is, as expected, the interarrival time of the events of a

Poisson process with time-dependent frequency of event occurrence [23].

In the following we analyze some examples of systems for which expressions of $p_\tau(\tau)$ can be found analytically, first studying processes in which the dynamics between fires is driven only by a deterministic drift and then also adding a diffusive term. The fire frequency will be considered to depend only on $x(t)$ and not also to be a direct function of time.

III. PROCESSES WITHOUT DIFFUSION

We start by studying processes described by Eq. (1) with $g(x)=0$. Since the fires always reinitialize the system to zero, for the system to leave $x=0$ after a renewal event, the condition $f(0)\neq 0$ must also be satisfied. In these cases, the trajectories followed by $x(\tau)$ between fires are deterministically defined by a one-to-one relation between τ and x—i.e., $\tau=\int_{x_0}^{x} du/f(u)$, where τ is either the time from the beginning of the process or the time after any given fire or renewal event (when $x_0=0$). Therefore, given the assumption that the renewal events occur according to a Poisson process, the PDF of the times between fires can be written as [23]

$$p_\tau(\tau)=\lambda(x(\tau))\exp\left[-\int_0^\tau \lambda(x(u))du\right], \tag{10}$$

which is a function of $\lambda(x)$, $f(x)$, and x_0. According to the theory of renewal processes [10], the age-specific failure rate is given by the ratio $p_\tau(\tau)/\mathcal{F}(\tau)$, which coincides with $\lambda(\tau)$.

The same result can also be obtained through Eq. (6). In fact, the solution of Eq. (3) when the diffusive term is not considered is

$$p_\lambda(x,\tau)=\frac{1}{f(x)}\delta\left[\int_{x_0}^{x}\frac{du}{f(u)}-\tau\right]\exp\left\{-\int_0^\tau \lambda(x(t))dt\right\}, \tag{11}$$

which represents an atom of probability with decreasing mass moving along the curve $\tau=\int_{x_0}^{x} du/f(u)$, x_0 being the initial condition. Substituting this solution into Eq. (6) one obtains Eq. (10), which (we recall) represents the PDF of the time to the occurrences of the first fire for a system starting from x_0 as well as the PDF of interfire times when x_0 is equal to zero.

In the cases with no diffusion and in stationary conditions, $p_\tau(\tau)$ can be obtained following two other approaches, the analysis of which is useful to better understand the properties of these processes.

(i) Under steady-state conditions (i.e., for $x_0=0$), $p_\tau(\tau)$ can be derived directly from the solution of Eq. (2), $p(x)$. Assuming, without loss of generality, that $f(x)\geq 0$ when x is positive, the steady-state PDF satisfies

$$-\frac{d}{dx}[f(x)p(x)]-\lambda(x)p(x)+\delta(x)\langle\lambda\rangle=0, \tag{12}$$

the general solution of which reads

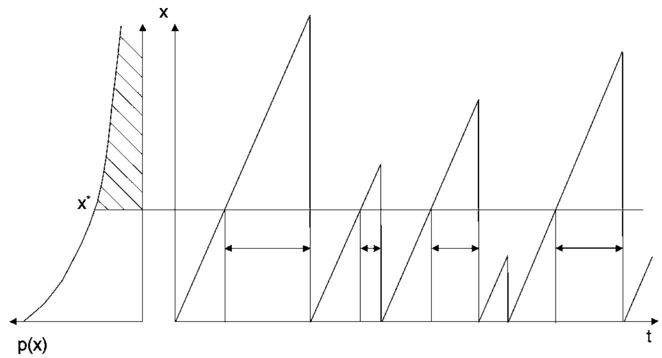

FIG. 1. Schematic example showing the relation between the area subtended by $p(x)$ and the survivor function (see text for details).

$$p(x)=\theta(x)\frac{\langle\lambda\rangle}{f(x)}\exp\left[-\int_0^x\frac{\lambda(u)}{f(u)}du\right]\quad (x\geq 0), \tag{13}$$

where $\theta(\cdot)$ is the Heaviside function and the value of $\langle\lambda\rangle$, which is not known *a priori*, can be obtained imposing $\int_0^{+\infty}p(x)dx=1$. To find the relation between $p(x)$ and $p_\tau(\tau)$, we can consider the portions of trajectory between fires as independent realizations of the same renewal process. In fact, as shown in Fig. 1, the probability that $x\geq x^*$ corresponds to the probability of the residual lengths of the trajectories that last for a time longer that τ^* (i.e., forward recurrence time [10]). Noticing that in stationary conditions the process becomes an equilibrium renewal process, according to [10] one can write

$$\int_{x^*}^{+\infty}p(x)dx=\int_{\tau^*}^{+\infty}\frac{\mathcal{F}(\tau)}{\langle\tau\rangle}d\tau, \tag{14}$$

from which

$$\mathcal{F}(\tau)=\langle\tau\rangle p(x)\frac{dx}{d\tau}=\frac{p(x)f(x)}{\langle\lambda\rangle}. \tag{15}$$

Substituting into Eq. (13) and using Eq. (5), Eq. (10) is readily obtained.

(ii) An alternative interpretations of the process of deterministic fire growth and successive renewals can be given by assuming that $x(t)$ follows a deterministic trajectory according to $f(x)$ up to a random threshold, the value of which is extracted by a distribution $p_f(x)$. When x reaches such a threshold, a fire is released and the process starts again until it reaches a new threshold that is again extracted from the distribution $p_f(x)$ and so on. The distribution of the thresholds can be derived by $p_\tau(\tau)$ [Eq. (10)] as $p_f(x)=p_\tau(x(\tau))(d\tau/dx)=\lambda(x)p(x)/\langle\lambda\rangle$, provided that $\tau=\int_0^x du/f(u)$. This implies that the two distributions $p(x)$ and $p_f(x)$ coincide when λ is constant (i.e., equal to $\langle\lambda\rangle$) and thus they are exponential. A similar approach has been used in a simplified neuronal model [17,18], where the neuron excitations were assumed to be released when the voltage across the nerve membrane reaches random thresholds.

An example of a realization of a state-dependent fire process is reported in Fig. 2 summarizing the corresponding

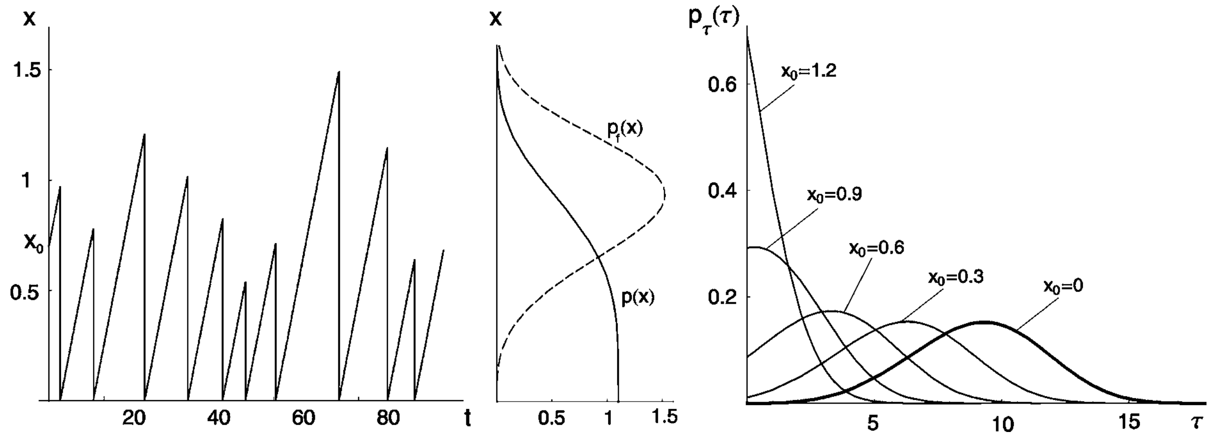

FIG. 2. Example of time series of x (left) in the case of constant drift, $k=0.1$, and $\lambda(x)=0.4x^3$. The steady-state PDF of x (solid line) and the PDF of the values of x at which a fire occurs (dashed line) are shown in the middle graph, while examples of PDF's of the times of first-fire occurrence for different values of x_0 and the interfire time PDF (i.e., $x_0=0$) are shown on the right.

related PDF's introduced before [e.g., $p(x)$, $p_f(x)$, and $p_\tau(\tau)$]. The system, which is a particular case of those studied in Sec. III A, is driven by a constant drift [e.g., $f(x)=0.1$ and $g(x)=0$] and has a cubic dependence of the mean fire rate of fire occurrence [e.g., $\lambda(x)\propto x^3$].

It is also interesting to note that when the system reaches a steady state, the jumps towards zero are balanced by the drift. Hence, in stationary condition $\langle x\lambda(x)\rangle=\langle f(x)\rangle$. Such a relation can be obtained either by multiplying Eq. (2) to x and integrating over the x-domain Ω or directly averaging Eq. (1) when $g(x)=0$. It follows that $\langle F(x,t)\rangle=\langle x\lambda(x)\rangle$. In fact, since the system jumps to zero when a fire is released, $\langle F(x,t)\rangle$ represents the average of the product between the mean rate of fire occurrence, $\lambda(x)$, and the corresponding value of the jumps, x.

In what follows we will discuss two families of processes driven by different forms of $\lambda(x)$ with constant positive drift—i.e., $f(x)=k$ with $k>0$.

A. Power-law dependence of λ on x

In many physical systems the probability of having a fire increases with the value of the state variable. Examples of such a behavior are represented by natural fires in ecosystems, the frequency of which depends on the amount of vegetation biomass [9], and neuron models, where fires are released when the voltage reaches or is around a certain threshold [17,18].

We will assume that $\lambda(x)$ is represented by a power law $k_1 x^\alpha$ ($\alpha \geqslant 0$), so that the probability of having a fire either remains constant (when $\alpha=0$) or increases as x grows. Given the constant drift, λ depends on time as $\lambda=k_1[k\tau+x_0]^\alpha$, as follows from Eq. (1). As a result, the system has positive aging [10]; i.e., the longer the time passed after a fire, the more likely is immediate firing. Moreover, because of this particular form of λ, $\langle F(x,t)\rangle=k_1\langle x^{\alpha+1}\rangle$, and the $(\alpha+1)$th moment of x can be evaluated directly from Eq. (1) as the ratio k/k_1.

The steady-state PDF of x and the PDF of the time between fires are [see Eqs. (13) and (10)]

$$p(x)=\frac{A}{k}\exp\left[-\frac{k_1}{k}\frac{x^{\alpha+1}}{\alpha+1}\right], \tag{16}$$

$$p_\tau(\tau)=k_1(k\tau)^\alpha\exp\left[-\frac{k_1}{k}\frac{(k\tau)^{\alpha+1}}{\alpha+1}\right], \tag{17}$$

where A is a normalization constant. Equation (17) can also be obtained from Eq. (6). In fact, in this case, $p_\lambda(x,\tau)$ reads

$$p_\lambda=\delta(x-k\tau-x_0)\exp\left[-\frac{k_1}{k(\alpha+1)}(x^{\alpha+1}-x_0^{\alpha+1})\right], \tag{18}$$

and from Eq. (6) one obtains

$$p_\tau(\tau)=k_1(x_0+k\tau)^\alpha\exp\left\{-\frac{k_1}{k(\alpha+1)}[(x_0+k\tau)^{\alpha+1}-x_0^{\alpha+1}]\right\}, \tag{19}$$

which represents the probability of the time of first-fire occurrence starting from $x(0)=x_0$. When $x_0=0$, Eq. (19) is the PDF of interfire times and, in fact, becomes equal to Eq. (17).

Figure 3 shows some distributions of both x and τ for different values of the parameter α. When $\alpha=0$, the frequency of the jumps is constant (equal to $1/k_1$) and both distributions are exponential. In general, for $\alpha>0$, $p_\tau(\tau)$ is a power-exponential distribution [26]. When $k_1=k(\alpha+1)$, $p_\tau(\tau)$ is a Weibull distribution, which is one of the asymptotic distributions of general extreme value theory and is commonly used for the statistics of lifetime of objects [27]. The Weibull distribution has also been adopted in the analysis of natural fire occurrence in ecosystems [21].

Interestingly, when α tends to $+\infty$, $\lambda(x)\to\delta(x-1)$, $p_\tau(\tau)=\delta(\tau-1/k)$, and $p(x)$ becomes a uniform distribution between 0 and 1. In this particular situation the process becomes totally deterministic, with $x(t)$ linearly increasing up to $x=1$ and reset to $x=0$ thereafter.

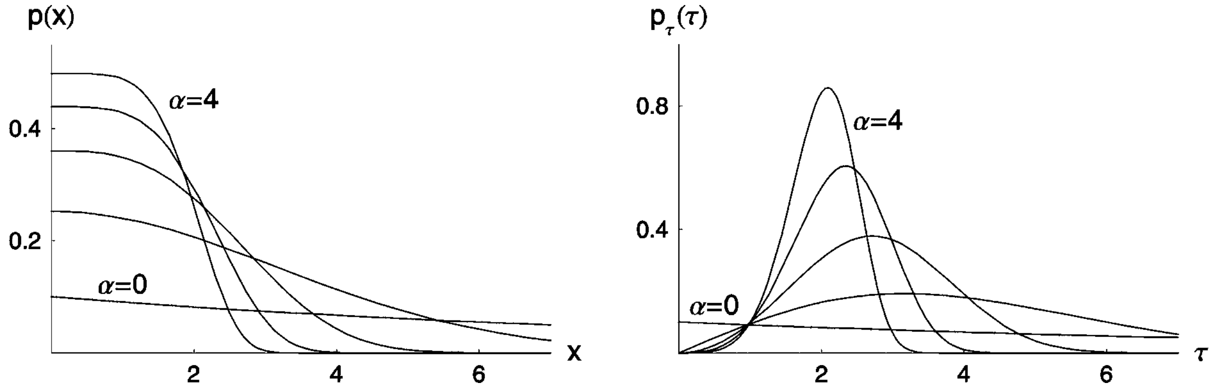

FIG. 3. Steady-state distributions of x and τ in the case of constant drift k and $\lambda = k_1 x^\alpha$, with α varying between 0 and 4 with step of 1. The other two parameters are $k=1$ and $k_1=0.1$.

The previous results are also valid for $-1<\alpha<0$, in which case the ageing is negative (i.e., the more the system lasts without a fire, the less likely is immediate fire occurrence). For these values of α, $p_\tau(\tau)$ becomes a stretched exponential [26].

B. Hyperbolic dependence of λ on *x*

In this second example $\lambda(x)$ will be assumed to have the form

$$\lambda(x) = \frac{a+bx}{c-dx}, \tag{20}$$

where a, b, and c are positive parameters, while d is a constant that might also be negative. This choice of the parameters allows $p(x)$ to be normalized [i.e., $\int_0^{+\infty} p(x)dx = 1$] and therefore the system can reach a stationary state. After Eqs. (13) and (10), the steady distribution of x and the PDF of the fire occurrences are

$$p(x) = \frac{A}{k} \exp\left[\frac{bx}{dk}\right] |c-dx|^{(ad+bc)/d^2k},$$

$$p_\tau(\tau) = \frac{a+bk\tau}{c-dk\tau} \exp\left[\frac{b\tau}{d}\right] \left|1-\frac{dk\tau}{c}\right|^{(ad+bc)/d^2k}, \tag{21}$$

with A the normalization constant.

Depending on the sign of d, either the value of x or that of λ is bounded.

When $d>0$, λ has a vertical asymptote in $x=c/d$, there is positive aging, and the fire frequency tends to $+\infty$ as x approaches c/d. If the initial value of x is comprised between 0 and c/d, the process remains in that interval of values. Figure 4 shows the stationary PDF of x and $p_\tau(\tau)$ for different b in the case of $a=0$, which implies that the aging function is 0 in $x=0$ and increases up to ∞ as x approaches c/d. Accordingly, the interarrival time between two fires is also bounded at $\tau=c/(kd)$. As b increases, the PDF of x moves towards zero, because the jumps are on average more frequent. For $b=0$ and $a\neq 0$ (not shown), the PDF of x becomes a power law in the interval $0<x<c/d$, while $p_\tau(\tau)$ is a generalized Pareto distribution bounded between 0 and $c/(kd)$ [27].

When $d<0$, the process can assume any value of the positive x axis, but λ has an horizontal asymptote, which can be attained from above (i.e., negative aging) or below (i.e., positive aging) depending on whether a/c is higher or lower than $-b/d$, respectively. Consequently, the probability that a fire is released tends to a constant value when x is far from

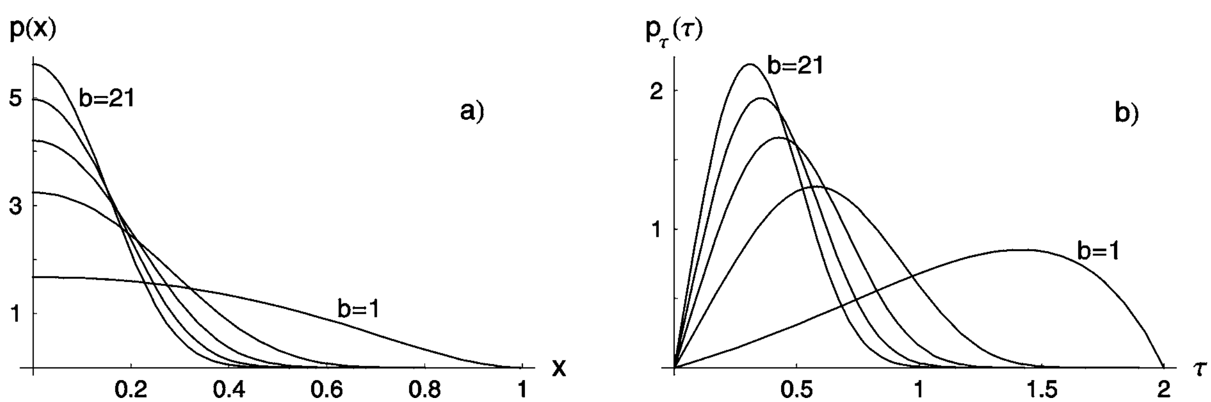

FIG. 4. Steady-state distributions of x and τ in the case of $\lambda=(a+bx)/(c-dx)$ when x is upper bounded ($a=0$, $c=1$, $d=1$, $k=0.5$, and b varies from 1 to 21 with steps of 5).

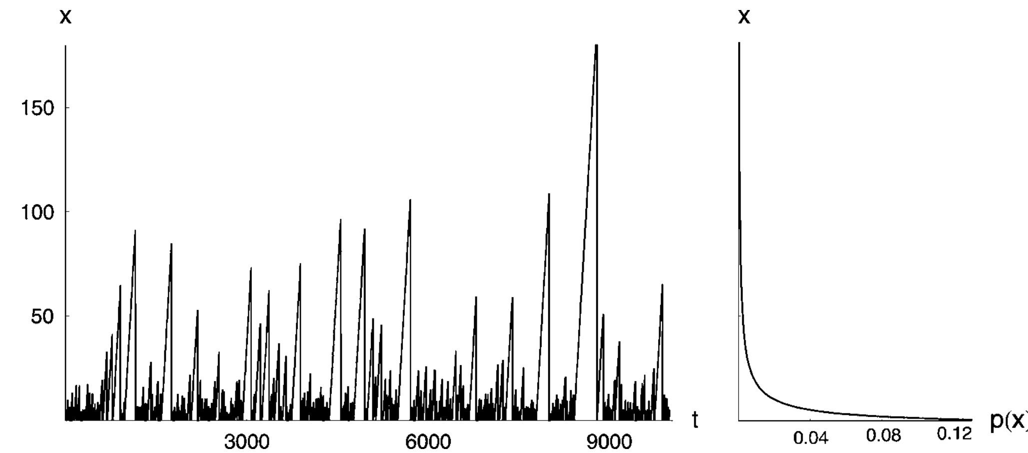

FIG. 5. Time series and corresponding power-law tail PDF of x in the case of $\lambda=(a+bx)/(c-dx)$ when x is unbounded, but λ is limited ($a=0.2$, $b=0$, $c=1$, $d=-1/4$, and $k=0.5$).

the origin. In particular, when $a/c=-b/d$, λ is constant and $p(x)$ and $p_\tau(\tau)$ are exponential distributions. It is interesting to note that, for $b\neq 0$, the PDF's of Eq. (21) are power exponentials, while, when $b=0$, they become generalized Pareto distributions, as shown in Fig. 5 [in such a case the condition $|a/(dk)|>1$ must hold to guarantee the decay to zero of $p(x)$ when $x\to\infty$]. The origin of power-law tails is due to the contrasting action of the fires that bring the system to zero and the drift, which repels the system from the origin [25,26]. Power laws have been recently found in timing of human actions [28–30], for which the present approach of state-dependent Poisson processes might provide a simple mathematical description.

IV. PROCESSES WITH DIFFUSION

In many cases, random external fluctuations affect the dynamics during interfire periods. These effects are commonly represented by either additive or multiplicative Gaussian white noise, through a suitable choice of the function $g(x)$ in Eq. (1). Numerous examples of these types of models can be found in the literature, such as the already mentioned leaky integrate-and-fire model in neurobiology [14,15] or the stochastic logistic equation in population biology [3]. In the case of natural fires in ecosystems, the growth of vegetation biomass has unpredictable components due to hydroclimatic variability [25], which might be modeled as a Gaussian noise.

In the following we analyze three examples of fire models with Gaussian noise for which analytical solutions can be obtained.

A. Absorbing barrier

A special limiting case of this type of processes can be found in the theory of the first-passage time through an absorbing barrier at a constant value, $x=a$. This is equivalent to assuming that the state-dependent frequency of fires is $\lambda(x)=\delta(x-a)$, so that the system evolves following its trajectory until it reaches the value $x=a$, at which a renewal event occurs. Clearly, the intertime distribution of the fire occurrences is equal to the PDF of the first-passage time through an adsorbing barrier in $x=a$ [5,14,23] and Eq. (5) coincides with the first-passage time through $x=a$.

An example of these type of processes was used in [14] to describe neuronal activity. In this model, the voltage across a nerve membrane, x, is assumed to evolve according to a Wiener process and to be reinitialized by means of a fire every time the voltage reaches the threshold value $x=a$. The resulting PDF of fire occurrences is the well-known inverse Gaussian distribution with power-law decay, $t^{-3/2}$ [14,23].

B. Additive noise

In this subsection we study processes with no drift [i.e., $f(x)=0$] forced by additive Gaussian noise [i.e., $g(x)=1$] and fires occurring with frequency $\lambda(x)=k_1x^2$. The process is thus a random walk between fires, the occurrence of which becomes more probable as $x(t)$ moves far from the origin.

Solving Eq. (2) under steady-state conditions, it is possible to show that the system reaches a steady state in which the PDF of x is [31]

$$p(x)=\frac{(k_1/\sigma^2)^{3/8}}{2\sqrt{2\pi}\Gamma(5/4)}\sqrt{|x|}K_{1/4}\left(\frac{\sqrt{k_1}x^2}{2\sqrt{\sigma^2}}\right), \tag{22}$$

where $\Gamma(\cdot)$ is the gamma function and $K_n(\cdot)$ is the modified Bessel function of the second kind [32]. The PDF of x decays exponentially for $|x|\to+\infty$ and becomes more concentrated around $x=0$ as k_1 increases, while it tends to be more spread out for high values of σ^2.

To study the interfire statistics we adopt Eq. (3), which reads, in this case,

$$\frac{\partial}{\partial\tau}p_\lambda(x,\tau)=\sigma^2\frac{\partial^2}{\partial x^2}p_\lambda(x,\tau)-k_1x^2p_\lambda(x,\tau), \tag{23}$$

with initial condition $p_\lambda(x,0)=\delta(x-x_0)$. The solution of Eq. (23) can be written as [33]

$$p_\lambda(x,\tau)=\frac{N(x,\tau)}{D(x,\tau)}, \tag{24}$$

where

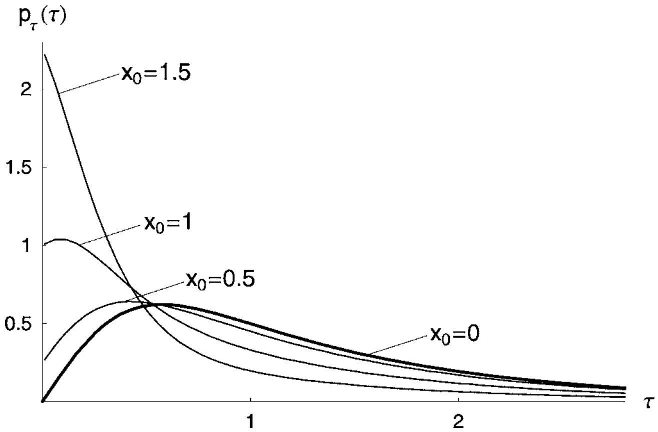

FIG. 6. Examples of PDF's of the times of first-fire occurrence for different values of x_0 and the interfire time PDF (i.e., $x_0=0$, thick line) for a process driven by additive Gaussian white noise between fires occurring at a rate $\lambda=k_1x^2$ (Sec. IV B). Parameters are $\sigma^2=1$ and $k_1=1$.

$$N(x,\tau)=\exp\left\{\sqrt{\sigma^2k_1}\tau+\frac{1}{2}\sqrt{\frac{k_1}{\sigma^2}}(x^2-x_0^2)+\sqrt{\frac{k_1}{\sigma^2}}\frac{[x\exp(2\sqrt{\sigma^2k_1}\tau)-x_0]^2}{1-\exp(4\sqrt{\sigma^2k_1}\tau)}\right\} \tag{25}$$

and

$$D(x,\tau)=\sqrt{\pi}\sqrt{\sqrt{\frac{\sigma^2}{k_1}}[\exp(4\sqrt{\sigma^2k_1}\tau)-1]}. \tag{26}$$

The PDF of the fire occurrences can be found inserting Eq. (24) into Eq. (4) and then substituting in Eq. (5). When $x_0\neq 0$, $p_\tau(\tau,x_0)$ cannot be obtained exactly, while for $x_0=0$ it reads

$$p_\tau(\tau)=\frac{\sqrt{2\sigma^2k_1}\exp(\sqrt{\sigma^2k_1}\tau)}{\sqrt{\exp(4\sqrt{\sigma^2k_1}\tau)-1}[\coth(2\sqrt{\sigma^2k_1}\tau)]^{3/2}}. \tag{27}$$

It is interesting to note that the PDF of the times between fires depends exclusively on the product σ^2k_1, since the intensity of the two random terms (i.e., diffusion and jumps due to fires) affects in the same way the system. In fact, high values of k_1 increase the probability of having a renewal event and, analogously, a large σ^2 tends to move the system far from the origin, leading to a higher probability of fire.

Figure 6 shows some PDF's of the times of first-fire occurrence for different values of x_0. Apart from that corresponding to $x_0=0$, the other curves are obtained by numerical integration of Eq. (24) with respect to x. Given the dependence of λ on x^2, it is apparent that higher values of the initial state lead to higher probability to have fires in short times.

Figure 7(a) shows different $p_\lambda(x,t)$ at different times t with initial condition $x_0=1$. The PDF starts from a Dirac δ at $\tau=0$ and then spreads out and moves towards the origin because of the effect of both k_1 and σ. As τ increases, the area of p_λ decreases, going to zero when $\tau\to+\infty$. Figure 7(b) shows different distributions of times between fires (in this case $x_0=0$). It is clear that increasing the product σ^2k_1 induces more frequent fires.

Since in this case there is not a one-to-one relation between x and τ, the function $\lambda(x(\tau))$ cannot be written explicitly as a function of time. However, we can find an equivalent age-specific failure rate, $\tilde{\lambda}(\tau)$, using the relation $\tilde{\lambda}(\tau)=p_\tau(\tau)/\mathcal{F}(\tau)$ [10]. According to Eq. (27), one obtains

$$\tilde{\lambda}(\tau)=\sqrt{k_1\sigma^2}\tanh(2\sqrt{k_1\sigma^2}\tau), \tag{28}$$

showing that the system has positive aging and its age-specific failure rate grows from zero for $\tau=0$ to $\sqrt{k_1\sigma^2}$ when $\tau\to+\infty$ (Fig. 8). Therefore, as the time from the last fire increases, the renewal process behaves as a Poisson process.

C. Multiplicative noise

In this subsection we study systems with $g(x)=\sqrt{x}$, $\lambda(x)=k_1x$ and with no drift [i.e., $f(x)=0$]. For these processes $x(t)$ is always positive and, as $x(t)$ moves closer to the origin, the strength of the Gaussian noise and the frequency of fires are reduced. Moreover, after a fire the system jumps to $x(t)=0$ and remains there, because of the form of $g(x)$ and $\lambda(x)$, which are both zero in the origin. Thus only one fire can occur. As will be discussed more in detail in the following, it is also possible that the system reaches the state $x(t)=0$ before a fire takes place [6,7]. In such a case no fire will occur,

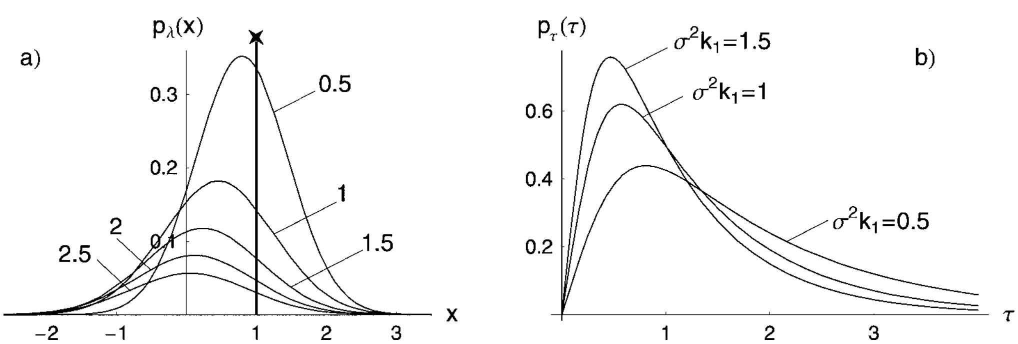

FIG. 7. Probability distributions of a process driven by additive Gaussian noise and fires with $\lambda=k_1x^2$ (Sec. IV B). (a) $p_\lambda(x,t)$ at different times t, starting from $x_0=1$ at $t=0$ to $t=2.5$ with $\sigma^2=0.5$ and $k_1=1$. (b) Probability distribution of the times between fires ($x_0=0$) for different values of σ^2k_1.

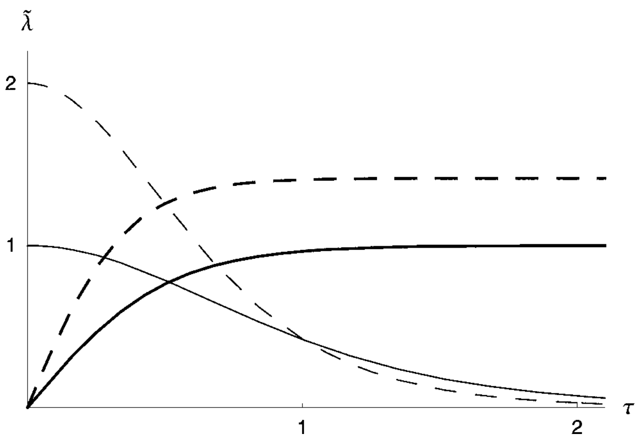

FIG. 8. Aging functions $\tilde{\lambda}(\tau)$ for the two processes studied in the Secs. IV B (thick lines, corresponding to positive aging) and IV C (thin lines, corresponding to negative aging). Solid lines have $k_1=1$ and dashed lines $k_1=2$. Other parameters are $x_0=1$ and $\sigma^2=1$.

since the system remains at zero after it reaches that state.

The PDF of the times of first-fire occurrence can be obtained from the solution of Eq. (3). According to the chosen forms of $g(x)$ and $\lambda(x)$, Eq. (3) reads

$$\frac{\partial}{\partial \tau}p_\lambda(x,\tau) = \sigma^2\frac{\partial^2}{\partial x^2}[xp_\lambda(x,\tau)] - k_1 x p_\lambda(x,\tau), \qquad (29)$$

with initial condition $p_\lambda(x,0)=\delta(x-x_0)$ $(x_0\neq 0)$. Laplace-transforming Eq. (29) yields

$$\frac{\partial}{\partial \tau}p_\lambda^*(u,\tau) + (\sigma^2u^2 - k_1)\frac{\partial}{\partial u}p_\lambda^*(u,\tau) = 0, \qquad (30)$$

the solution of which can be obtained with the method of the characteristics [5,34] as

$$p_\lambda^*(u,\tau) = \exp\left\{-\frac{x_0\sqrt{k_1}}{\sqrt{\sigma^2}}\tanh[\sqrt{k_1\sigma^2}\tau + \tanh^{-1}(u\sqrt{\sigma^2/k_1})]\right\}. \qquad (31)$$

According to Eq. (7), the survivor function is

$$\mathcal{F}(\tau) = \exp\left[-\frac{x_0\sqrt{k_1}}{\sqrt{\sigma^2}}\tanh(\sqrt{k_1\sigma^2}\tau)\right], \qquad (32)$$

which is a function decreasing from 1 at $\tau=0$ to $\exp(-x_0\sqrt{k_1/\sigma^2})$ when τ tends to $+\infty$. The finite value attained by $\mathcal{F}(\tau)$ for large τ accounts for those cases when the system reaches the state $x(t)=0$ before a fire is released and gives the probability of having no fires. As $\sigma\to 0$, $\mathcal{F}(\tau)$ goes to zero for large τ. In fact, for $\sigma\to 0$, $x(t)$ tends to linger around the initial value x_0 and the distribution of the interfire times tends to an exponential with average k_1x_0, since the process becomes equivalent to a Poisson process with constant frequency of fire occurrence equal to k_1x_0.

According to the survivor function [e.g., Eq. (4)], the PDF of the interfire times reads

$$p_\tau(\tau) = x_0k_1[\text{sech}(\sqrt{k_1\sigma^2}\tau)]^2\exp\left[-\frac{x_0\sqrt{k_1}}{\sqrt{\sigma^2}}\tanh(\sqrt{k_1\sigma^2}\tau)\right], \qquad (33)$$

the area of which is $1-\exp(-x_0\sqrt{k_1/\sigma^2})$, because of a mass equal to $\exp(-x_0\sqrt{k_1/\sigma^2})$ that accounts for the probability of having no fire.

Also in this case it is possible to evaluate an equivalent age-specific failure rate as $\tilde{\lambda}(\tau)=p_\tau(\tau)/\mathcal{F}(\tau)$, which according to Eqs. (32) and (33) reads

$$\tilde{\lambda}(\tau) = x_0k_1[\text{sech}(\sqrt{k_1\sigma^2}\tau)]^2, \qquad (34)$$

which shows that the system has a negative aging, since the frequency that the fire occurs diminishes in time (Fig. 8).

V. CONCLUSIONS

We have presented a study about the statistical dynamics of processes driven by fires with state-dependent rate of occurrence, which reinitialize the system to zero. The links between the fire occurrence and the dynamics of these state-dependent renewal processes are formally derived for different processes.

Probability distributions of the state variable $x(t)$ and fire occurrences are obtained exactly for general forms of drifts $f(x)$ and age-specific failure rates $\lambda(x)$. Power-law tail distributions of the state variable and interfire times are analyzed in detail for processes with constant drift coupled to fires with frequency $\lambda(x)\propto 1/(a+x)$ $(a>0)$ (see Fig. 4). A few applications to processes driven by Gaussian noise for which exact results can be obtained are also shown. In particular, two processes with opposite aging are studied. The first process, driven by additive Gaussian noise with intensity σ^2 between fire events that occur with state-dependent mean rate $\lambda(x)=k_1x^2$, presents positive aging depending on the product $k_1\sigma^2$ [see Eq. (27)]. As shown in Fig. 8, the frequency of fire occurrence increases in time and reaches a constant value, so that the process tends to behave as a Poisson process when τ is large. The dynamics of the second process is driven by the action of Gaussian white noise with strength σ^2 modulated by the function $g(x)=\sqrt{x}$ and fires occurring at a mean rate $\lambda(x)=k_1x^2$. Such a process has negative aging; i.e., $\tilde{\lambda}$ decreases with τ (Fig. 8), since once the process reaches values close to zero it tends to remain there (because of the multiplicative nature of the Gaussian noise) with a reduction in the probability of fire occurrence.

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